Probability 1 hmwk 1

All problems are from Gordon's webpage. (Maybe I will include a pdf copy of the questions later)

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1.1 1

Let $S = \{1, \dots, n\}$. Note that the number of different partitions on the S and the number of different algebras on this set are both finite. First I will show that the number of different partitions on S is $\leq$ the number of algebras on S.

Given an algebra, $\mathscr{A}$ we can construct a partition in the following way. Step 1: Let $x_1 = 1$ and define the set

$$A_{x_1} = \bigcap_{x_1 \in A, A \in \mathcal{A}} A =: \text{The smallest set in } \mathscr{A} \text{ containing } x_1$$

Step 2: Let x_2 be the smallest number such that $x_2 \notin A_{x_1}$. Define the set

$$A_{x_2} = \bigcap_{x_2 \in A, A \in \mathscr{A}}$$

Step 3: Repeat this process to generate the sets $A_{x_1}, A_{x_2}, \dots, A_{x_k}$ (where x_i is the smallest number s.t. $x_i \notin A_{x_j}$, $j \leq i$). Notice that this process will eventually terminate because the set S is finite and the number of elements in S but not in a set A_{x_i} decreases by at least one with each step.

Lemma 1. *The sets $A_{x_1}, A_{x_2}, \dots, A_{x_k}$ form a partition of S.*

Proof. It is clear by construction that every $x \in S$ appears in some A_{x_i}. Hence it is enough to show that if $i \neq j$ then $A_{x_i} \cap A_{x_j} = \emptyset$. Suppose that $A_{x_i} \cap A_{x_j} \neq \emptyset$. Then the set $A_{x_i} \cap A_{x_j} \in \mathscr{A}$. We will break into cases.

Case: $x_i \in A_{x_i} \cap A_{x_j}$. If $A_{x_i} \subseteq A_{x_j}$ then we can produce a set containing x_j which is strictly smaller than A_{x_j} to contradict the minimality of A_{x_j}. (If $x_j \in A_{x_i}$ then take $A_{x_i} \cap A_{x_j}$ and if $x_j \notin A_{x_i}$ then take $A_{x_j} \setminus A_{x_i}$). If $A_{x_i} \not\subseteq A_{x_j}$ then $A_{x_i} \setminus A_{x_j} \neq \emptyset$. This implies that the set $A_{x_i} \cap A_{x_j}$ is of size strictly smaller then A_{x_i}, which contradicts the minimality of A_{x_i}.

Case: $x_i \notin A_{x_i} \cap A_{x_j}$. In this case the set $A_{x_i} \setminus A_{x_j}$ is strictly smaller than A_{x_i}, which contradicts the minimality of A_{x_i}.

Since in either case we get a contradiction it must be that $A_{x_i} \cap A_{x_j} = \emptyset$. $\square$

Lemma 2. *The map which produces a partition from an algebra is injective. That is, given algebras* $\mathscr{A}_1 \neq \mathscr{A}_2 \implies \mathscr{P}(\mathscr{A}_1) \neq \mathscr{P}(\mathscr{A}_2)$.

Proof. I will argue that the process of generating an algebra from a partition is an inverse map to my process of creating a partition from an algebra. That is, I will show for a given algebra $\mathscr{A}$,

$$\mathscr{A} = \mathscr{A}(\mathscr{P}(\mathscr{A}))$$

First it is clear that $\mathscr{A} \supseteq \mathscr{A}(\mathscr{P}(\mathscr{A}))$ because the partition $\mathscr{P}(\mathscr{A}) \subset \mathscr{A}$. Hence it is enough to show $\mathscr{A} \subseteq \mathscr{A}(\mathscr{P}(\mathscr{A}))$. Given a set $B \in \mathscr{A}$, let $\bigcup_i A_{x_i}$ be a cover of B by sets A_{x_i} which are from the generated partition, $\mathscr{P}(\mathscr{A})$. By construction $B \subseteq \bigcup_i A_{x_i}$. Next I argue that $B \supseteq \bigcup_i A_{x_i}$. Suppose for a contradiction that there exists a $y \in \bigcup_i A_{x_i} \setminus B$. Since this is a disjoint union, it must be that y is in a single set of $\bigcup_i A_{x_i}$. Call this set $y \in A_{x_k}$. Then depending on whether $x_k \in B$ or $x_k \notin B$ we could a produce a set containing x_k which is smaller than A_{x_k} (by taking either intersection with B or set minus with B resp.). This would contradict the minimality of A_{x_k}. Therefore it must be that no such y can exist and we get that $\bigcup_i A_{x_i} \setminus B = \emptyset \implies \bigcup_i A_{x_i} \subseteq B$.

Hence $B = \bigcup_i A_{x_i}$ and so $B \in \mathscr{A}(\mathscr{P}(\mathscr{A}))$. We can conclude the claim. $\square$

Claim 1. *The number of different partitions on the set* $\{1, \ldots, n\}$ *is* $\geq$ *the number of algebras on this set.*

Proof. By the above lemmas we have an injective map from algebras on S to partitions on S. $\square$

Next I will show that the number of different partitions on the set $\{1, \ldots, n\}$ is $\leq$ the number of algebras on this set.

Lemma 3. *Given a partition* $\mathscr{P}$ *of* S *every set* $A \in \mathscr{A}(\mathscr{P})$ *is the disjoint union of sets from* $\mathscr{P}$.

Proof. Let $\mathscr{P} = A_1 \sqcup \cdots \sqcup A_k$. I will prove the claim by induction on the number of union and complement operations used to a produce a set in $\mathscr{A}(\mathscr{P})$. Base case: Define the set $B_{1,1} = \{$ all sets that can be made from unions of sets in $\mathscr{P}\}$ and the set $B_{1,2} = \{$ all sets that can be made from complementing sets in $\mathscr{P}\}$. Let $B_1 = B_{1,1} \cup B_{1,2}$. By construction every set in $B_{1,1}$ is a disjoint union of sets in $\mathscr{P}$ and because the sets $A_1, \ldots, A_k$ are disjoint the complement of A_i is a disjoint union of sets in $\mathscr{P}$ too. Hence B_1 has every set in it being a disjoint union of sets in $\mathscr{P}$.

Next define a set $B_{2,1} = \{$ all sets made from unions of sets in $B_1\}$ and $B_{2,2} = \{$ complements of sets in $B_1\}$. Let $B_2 = B_{2,1} \cup B_{2,2}$. In a similar process we can inductively construct sets B_n from prior B_{n-1} sets.

Inductive step: Assume that B_n has the property that every set in B_n is a disjoint union of sets in $\mathscr{P}$ and show that B_{n+1} also has this property. First see that $B_{n+1,1}$ has this property since the union of twos sets which are disjoint unions of sets in $\mathscr{P}$ will still be a disjoint union of sets in $\mathscr{P}$. Further $B_{n,2}$ also holds this property because a set in B_n may be written as a disjoint union of sets in $\mathscr{P}$ but $\mathscr{P}$ is partition, so the complement of this may still be written as a disjoint union of sets in $\mathscr{P}$. Hence the property holds for B_{n+1} too. □

Lemma 4. *The map which sends partitions of S to algebras of S is injective. That is if partitions $\mathscr{P}_1 \neq \mathscr{P}_2$ then $\mathscr{A}(\mathscr{P}_1) \neq \mathscr{A}(\mathscr{P}_2)$.*

Proof. Let $\{A_1, \ldots, A_k\} = \mathscr{P}_1$ and $\{B_1, \ldots, B_l\} = \mathscr{P}_2$. Suppose that $\mathscr{A}(\mathscr{P}_1) = \mathscr{A}(\mathscr{P}_2)$. Then for any $1 \leq i \leq l, B_i \in \mathscr{A}(\mathscr{P}_1)$ and by the previous lemma $B_i = \bigsqcup A_j$ for some sets A_j. Hence we have that $\mathscr{P}_1$ is a finer partition than $\mathscr{P}_2$. By a similar argument we can say that $\mathscr{P}_2$ is a finer partition than $\mathscr{P}_2$. Therefore it must be that under these assumptions $\mathscr{P}_1 = \mathscr{P}_2$. □

Claim 2. *The number of different partitions on the set $\{1, \ldots, n\}$ is $\leq$ the number of algebras on this set.*

Proof. By the prior claim there is an injective map from partitions on S to algebras on S. □

Claim 3. *The number of different partitions on the set $\{1, \ldots, n\}$ = the number of algebras on this set.*

Proof. By previous claims we have that the number of different partitions on $S \leq$ the number of algebras on S and the number of different partitions on $S \geq$ the number of algebras on S. Since these are both finite sets we have equality. □

1.2

Claim 4. *There are 5 algebras that can be made on the set $\{1, 2, 3\}$*

Proof. By the previous problem, counting the number of algebras on this finite set is equivalent to counting the number of partitions on this set. By trial and error one can write down all the partitions as so:

$$\{\{1,2,3\}\} \qquad \{\{1\}\{2,3\}\}$$
$$\{\{1,2\}\{3\}\} \qquad \{\{1,3\}\{2\}\}$$
$$\{\{1\}\{2\}\{3\}\}$$

□

1.3

Claim 5. *Any algebra $\mathscr{A}$ on a finite set has $|\mathscr{A}| = 2^k$ for some $k \in \mathbb{N}$.*

Proof. Recall from part (i) that given an algebra $\mathscr{A}$, we can produce a partition $\mathscr{P} = A_1 \sqcup \cdots \sqcup A_k$ that generates $\mathscr{A}$. By a lemma in part (i), every set in $\mathscr{A}$ is a disjoint union of sets in $\mathscr{P}$. If there are k sets in $\mathscr{P}$ then the total number of sets that can be made via disjoint unions is 2^k. □

Claim 6. *Any algebra $\mathscr{A}$, on an infinite set either has $|\mathscr{A}| = \infty$ or $|\mathscr{A}| = 2^k$ for some $k \in \mathbb{N}$.*

Proof. Take a generating set $\mathscr{E}$ for $\mathscr{A}$ (there always exists a generating set because $\mathscr{A}$ generates itself trivially). Suppose that $|\mathscr{E}|$ is infinite. In this case by iteratively picking sets and using setminuses we can produce a countably infinite subcollection of disjoint sets. Hence $|\mathscr{A}| = \infty$. Suppose then that $|\mathscr{E}| < \infty$. In this case we can again iteratively pick sets and using setminuses, produce a disjoint collection of sets, $A_1, \dots, A_k$. By a similar argument to the above claim, we can show that $|\mathscr{A}| = 2^k$. □

Claim 7. *There does not exist an algebra with 754 elements.*

Proof. The prime factorization of $754 = 2 \cdot 13 \cdot 29$. This is not a power of 2 and so by the above two claims cannot be an algebra of neither a finite nor infinite set. □

1.4

I don't know why it's justified to take a Taylor series of the e^{e^x-1}. I have been considering applying Taylor's theorem but I don't know why I can't also apply Taylor's theorem to a function such as $e^{-1/x}$. See wikipedia:

https://en.wikipedia.org/wiki/Non-analytic_smooth_function

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Assuming that you can take a Taylor series of the function e^{e^x-1}

Claim 8. $e^{e^x-1} = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ *where a_n is defined in the earlier parts of this problem.*

Proof. I will show this by proving $a_n = \left(\frac{d^n}{dx^n} e^{e^x-1}\right)\Big|_{x=0}$. I will do this by strong induction. The base case $n = 0$ is clear because

$$\left(\frac{d^n}{dx^n} e^{e^x-1}\right)\Bigg|_{x=0} = e^{e^x-1}\Big|_{x=0} = 1$$

and $a_0 = 1$. Inductive step: Assume that $a_k = \left(\frac{d^k}{dx^k} e^{e^x - 1}\right)\Big|_{x=0}$ for $k \leq n$ and prove the identity holds for $n+1$. To do this, I will apply the Leibniz product rule

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}} e^{e^x-1} &= \frac{d^n}{dx^n} e^{e^x-1} \cdot e^x \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} e^{e^x-1} \cdot \frac{d^{n-k}}{dx^{n-k}} e^x \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} e^{e^x-1} \cdot e^x \\
&= \sum_{k=0}^{n} \binom{n}{k} a_k \cdot e^x \quad \text{(By Induction Hypothesis)}
\end{aligned}$$

It follows that

$$\frac{d^{n+1}}{dx^{n+1}} e^{e^x-1}\Big|_{x=0} = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot e^x\Big|_{x=0} = \sum_{k=0}^{n} \binom{n}{k} a_k$$

Notice that by part (iv) this is exactly the formula for a_{n+1}. □

1.5

Claim 9. $a_1 = 1, a_2 = 2, a_3 = 5$

Proof. This is clear by trial and error, and a_3 was calculated in part (ii). □

Claim 10. $a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k$

Proof. I will show this formula holds by showing a way to count a_{n+1}. Recall by part (i) that counting the number of algebras on $\{1, \ldots, n+1\}$ is equivalent to counting the number of partitions on this set. We can count the partitions on this set by examining what set $n+1$ can lie in. For example, we can first count the partitions which have $n+1$ in its own singleton set. There is $1 = \binom{n}{n}$ ways to put $n+1$ its own singleton set and then there are a_n ways to partition the remaining elements. The term $\binom{n}{n} a_n$ corresponds to the last term in the summation.

Next we can count all partitions that have $n+1$ paired with exactly one other element from the set $\{1, \ldots, n\}$. There are $n = \binom{n}{n-1}$ to pair $n+1$ with an element from $\{1, \ldots, n\}$. Afterwards there are a_{n-1} ways to partition the remaining the elements. The term $\binom{n}{n-1} a_{n-1}$ corresponds to the second to last term in the summation. Continuing this process we get to the second term in the summation, $\binom{n}{1} a_1$ which counts the number of partitions which pair $n+1$ with $n-1$ elements from $\{1, \ldots, n\}$. Finally, the first term in the summation, $a_0 = 1$, can correspond to the trivial partition which collects all of the set into a single subset. Essentially, the k represents how many remaining elements need to be partitioned after grouping $n+1$ with $n-k$ elements. □

2

2.1

Something weird is that a metric is defined as a function to $\mathbb{R}$ however if the $i(\cdot, \cdot) = -\infty$ then we have $2^{-\infty}$ which is undefined in the reals. So formally, this function was not well defined. Maybe $i(\cdot, \cdot)$ could be redefined as a piecewise function to handle this.

Claim 11. *$d(s^1, s^2) = 2^{-i(s^1, s^2)}$, where $i(s^1, s^2) = \inf\{i \in \mathbb{N} : s_i^1 \neq s_i^2\}$ is a metric on $\{-1, 1\}^{\mathbb{N}}$.*

Proof. We need to show the 3 axioms for a metric. First, symmetry is clear from the definition. Second positivity is also clear. If s^1 and s^2 are different then suppose they differ at an index n. Then $d(s^1, s^2) \geq 2^{-n} > 0$ and if $s^1 = s^2$ then $i(s^1, s^2) = \inf\{\emptyset\} = \infty$ by definition. Then $d(s^1, s^2) = 2^{-\infty} = 0$. Finally, I need to show the triangle inequality. Given $s_1, s_2, s_3 \in \{-1, 1\}^{\mathbb{N}}$ I want to show that

$$d(s^1, s^2) \leq d(s^1, s^3) + d(s^2, s^3)$$

Consider the case that some two of the 3 elements are equal. If $s^1 = s^2$ then the inequality holds by positivity of d. If s^3 equals one of the other two sets, without loss of generality say this is s^1, then this becomes an equality. So finally suppose that no two elements are equal, that is $s^1 \neq s^2 \neq s^3$.

Let

$$\begin{aligned} i(s^1, s^2) &= n \\ i(s^2, s^3) &= m \\ i(s^1, s^3) &= k \end{aligned}$$

Note that s^1 and s^2 must agree on the indices before $\min\{m, k\}$ because they both match s^3 before $\min\{m, k\}$. Therefore $n \geq \min\{m, k\}$ (in fact $n = \min\{m, k\}$). But if $n = \min\{m, k\}$ then we have either $n = m$ or $n = k$ and it follows that

$$2^{-n} \leq 2^{-m} + 2^{-k}$$

This thus shows that

$$d(s^1, s^2) \leq d(s^1, s^3) + d(s^2, s^3)$$

□

2.2

Claim 12. $\{-1,1\}^{\mathbb{N}}$ *is compact under the metric* d.

Proof. Recall that in a metric space compactness is equivalent to limit point compactness, so I will show limit point compactness. Given a set $S \subseteq \{-1,1\}^{\mathbb{N}}$ there must exist at least one infinite subset of sequences which have a 1 in the first index or an infinite subset of sequences which have a -1 in the first index. Call this infinite subset S^1.

Similarly, there exists at least one infinte subset of S_1 of sequences that all have a 1 in index 2 or a subset of sequences that all have -1 in index 2. Call this set $S_2 \subset S_1$.

Repeat this process to iteratively produce an S_n for $n \in \mathbb{N}$.

Finally consider $\bigcap_{n\in\mathbb{N}} S_n$. This set is non-empty by construction and take $s \in \bigcap_{n\in\mathbb{N}} S_n$. That is at each index n, s has the appropriate value at index n so that $s \in S_n$. I will show that s is a limit point of S.

Given any $\epsilon > 0$ there exists an n s.t. $2^{-n} < \epsilon$. Then the ball $B(s, 2^{-n}) \cap S \neq \emptyset$. This is because if the intersection was empty it would mean there are no sequences in S which match s on the first n indices. However, by construction the set S_n is an infinte subset of S of sequences that must match s on the first n indices. Therefore any of the elements in S_n will land in $B(s, 2^{-n})$ and there are an infinte number of such elements. Since any ball of s intersects S non-trivially, I have show that s is a limit point of S. □

2.3

Let Γ be an indexing set to collection of $\{-1,1\}$ sets that make up the product.

Given any cylinder $C = \{s \in \{-1,1\}^{\mathbb{N}} : s \in B\}$ where B is some condition such that there exists $\gamma_1, \ldots, \gamma_n \in \Gamma$ with $B \subseteq \{-1,1\}^{\mathbb{N}}_{\gamma_1} \otimes \cdots \otimes \{-1,1\}^{\mathbb{N}}_{\gamma_n}$ I will show that C is both open and closed in the metric topology induced by d (as defined in part (i)).

Claim 13. *C is open.*

Proof. Given $s \in C$, consider $m = \max\{\gamma_1, \ldots, \gamma_n\}$. Then take a ball of radius 2^{-m} around s. Call this $B(s, 2^{-m})$. Then for any $s' \in B(s, 2^{-m})$ we have that s' and s agree on at least the first m indices. Since these m indices contain the indices of $\gamma_1, \ldots, \gamma_n$ it follows that s' must also be in B (because B is only conditioned on the this finite number of indices). Hence we have shown that $B(s, 2^{-m}) \subseteq C$. Since any point of C is in an open ball contained in C, C must be open. □

Claim 14. *C is a closed set.*

Proof. I will show that C^c is an open set by a similar method to the one used above. Let $m = \max\{\gamma_1, \ldots, \gamma_n\}$ as before and given $s \notin C$ consider the ball $B(s, 2^{-m})$. For any $s' \in B(s, 2^{-m})$ note that s' matches s on the first m indices. Since B is only conditioned on indices $\gamma_i \leq m, 1 \leq i \leq n$ it must be that $s' \notin B$ either. Therefore $B(s, 2^{-m}) \subseteq C^c$. Since every point of the complement lies in an open ball contained in C^c it follows that C^c is open. This shows that C is closed. □

2.4

Claim 15. *Each open ball is a cylinder.*

Proof. Given a ball $B(s,r)$, let $n = \inf\{k \in \mathbb{N} : 2^{-k} \geq r\}$. Then $B(s,r) = \{s' \in \{-1,1\}^{\mathbb{N}} : s'$matches s on the first n indices$\}$ and take this set to be the condition set for a cylinder set. More precisely, define a map

$$\begin{aligned} \phi : \{-1,1\}^{\mathbb{N}} &\longrightarrow \prod_{i=1}^{n} \{-1,1\}_i \\ s' &\longmapsto (\pi_1(s'), \dots, \pi_n(s')) \end{aligned}$$

Let $B = \{\phi(s)\}$ the singleton set and note that $B \in \bigotimes_{i=1}^{n} \{-1,1\}_i$ because each $\{-1,1\}_i$ is given the discrete σ-algebra and this is a finite n-tuple.

Then

$$B(s, 2^{-n}) = \{s' \in \{-1,1\}^{\mathbb{N}} : \phi(s') \in B\} = \{s' \in \{-1,1\}^{\mathbb{N}} : (\pi_1(s'), \dots, \pi_n(s')) \in B\}$$

And this last set is the definition of a cylinder set so the claim is shown. □

2.5

Claim 16. $\{-1,1\}^{\mathbb{N}}$ *is separable with respect to the metric* d.

Proof. I need to find a countable dense subset. Define a set

$$D_n = \prod_{i=1}^{n} \times \prod_{n+1}^{\infty} \{1\} = \{\text{all sequences witha tail of 1's beginning at index } n+1\}$$

Note $|D_n| = 2^n$ and define a set $D = \bigcup_{n=1}^{\infty} D_n$. The cardinality of D is countable because it is the countable union of countable sets. Further, I will show D is dense in $\{-1,1\}^{\mathbb{N}}$.

Given any $s \in \{-1,1\}^{\mathbb{N}}$ and $\epsilon > 0$, take $n \in \mathbb{N}$ so large s.t. $2^{-n} < \epsilon$. Then in the set D_n there will exist an element s' s.t. s' agrees with s on the first n indices (s' exists because D_n was constructed as the collection of all possible n prefixes). Hence $s\ inB(s', 2^{-n})$ and this shows that D is dense. □

2.6

Let $\mathcal{S}$ be the product σ-algebra of $\{-1,1\}^{\mathbb{N}}$ and let $\mathcal{B}$ be the Borel σ-algebra on $\{-1,1\}^{\mathbb{N}}$ with respect to the metric d.

Claim 17. $\mathcal{S} = \mathcal{B}$.

Proof. Recall from class that the product σ-algebra is the σ-algebra generated by cylinders and the Borel σ-algebra is generated by open sets (or in a metric space, equivalently open balls).

First note that $\mathcal{B} \subset \mathcal{S}$ because in a metric space the topology is generated by pen balls and by part (4) the open balls are cylinders and hence in $\mathcal{S}$.

Next I'll show $\mathcal{B} \supseteq \mathcal{S}$. By part (2) any cylinder, C, is an open set in the metric topology induced by d. Hence $C \in \mathcal{B}$ and it follows that $\mathcal{B} \supseteq \mathcal{S}$. $\square$

Remark 1. I think one can also prove that for any cylinder C it is open because $C = \bigcup_{d \in D \cap C} \bigcup_{n=1}^{\infty} B(d, 2^{-n})$. Where D is the countable dense subset found in part (5). (This makes me guess that part 2 might follow form part (2))

3

3.1

Recall that $\mathcal{L}^0$ is the space of measurable functions from S to $\mathbb{R}$ with the Borel σ-algebra.

Claim 18. $\{f = g\} = \{x \in S : f(x) = g(x)\} \in \mathcal{S}$

Proof. We have

$$\{f = g\} = \{x \in S : f(x) - g(x) = 0\} \qquad = (f-g)^{-1}(0)$$

But in the Borels of the reals, the singleton $\{0\}$ is measurable. By proposition 1.20 in the notes, $f - g$ is a measurable function and thus this is the preimage of a measurable set by a measurable function and hence it's in the σ-algebra $\mathcal{S}$. $\square$

Claim 19. $\{f < g\} = \{x \in S : f(x) < g(x)\} \in \mathcal{S}$

Proof. Similarly,

$$\{f < g\} = \{x \in S : f(x) - g(x) < 0\} \qquad = (f-g)^{-1}((-\infty, 0))$$

Again $f - g$ is measurable (by prop 1.20 in the online notes) and this is the preimage of an open ray $(-\infty, 0)$ which is a measurable set in the Borels of $\mathbb{R}$. Therefore the preimage is also in the σ-algebra $\mathcal{S}$. $\square$

3.2

I got a hint to look at the complement of the set G_f from online at:

`http://www.sosmath.com/CBB/viewtopic.php?f=23&t=55569`

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Claim 20. *The set* $G_f = \{(x, y) \in S \times \mathbb{R} : f(x) = y\}$ *is in the the* σ*-algebra of* $(S \times \mathbb{R}, \mathcal{S} \times \mathcal{B}(\mathbb{R}))$

Proof. Define "upper" and "lower" sets to the graph G_f. Call these U_f and L_f resp. defined as follows

$$U_f = \{(x, y) : y > f(x)\}, \quad L_f = \{(x, y) : y < f(x)\}$$

Notice that we can also write the following:

$$U_f = \bigcup_{q \in \mathbb{Q}} f^{-1}((-\infty, q)) \times (q, \infty), \quad L_f = \bigcup_{q \in \mathbb{Q}} f^{-1}((q, \infty)) \times (-\infty, q)$$

Notice that both the equalities here are due to the density of $\mathbb{Q}$. Further U_f and L_f are now seen as a countable union of measurable sets (they are measurable sets because they are products of measurable sets and hence elements in the product σ-algebra). Thus $U_f, L_f \in \mathcal{S} \times \mathcal{B}(\mathbb{R})$. Next see that $G_f^c = U_f \cup L_f$ and because σ-algebras are closed under complements, G_f, must be in the σ-algebra. □

Remark 2. Another grad student had a different way to do this. They defined a map $F : \mathbb{S} \times \mathbb{R} \to S \times \mathbb{R}$ as

$$F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ f(x) - y \end{pmatrix}$$

Then the see that $G_f = F^{-1}(S \times \{0\})$. All that is left is to show that F is a measurable map. So by results in class this is equivalent to showing that the composition with projection maps is measurable. So clearly $\pi_1 \circ F = Id : S \to S$ and this is the composition of measurable maps so it's clearly measurable. Then

$\pi_2 \circ F : S \times \mathbb{R} \to \mathbb{R}$ as $\pi_2 \circ F = f(x) - y$ but this also equals the map $F = f \circ \pi_1 - \pi_2$. This is the difference of two measurable maps and hence is measurable. Therefore G_f is the pre-image of the measurable map F.

4 Additional Problem

Claim 21. *The product σ-algebra is generated by product cylinders.*

Proof. First I will show that product cylinders are in the product σ-algebra. Given a product cylinder C,

$$C = \{x \in \prod X_i : \pi_i(x) \in B_i\}$$

where $i \in I$ some finite indexing set and B_i are elements in the σ-algebra on X_i. Then it is clear that

$$C = \bigcap_{i \in I} \pi_i^{-1}(B_i)$$

This is a finite intersection of the measurable sets and hence in the product σ-algebra.

Next I will show that the product σ-algebra is in the σ-algebra generated by product cylinders. Note that the product σ-algebra is generated by the sets $\pi_i^{-1}(B_i)$ for B_i in the σ-algebra of X_i. But this itself is a cylinder set because it may be described as

$$\pi_i^{-1}(B_i) = \{x \in \prod X_\alpha : \pi_i(x) \in B_i\}$$

This shows the claim. ∎

Claim 22. *I'd like to show that the product sigma algebra is generated by cylinders (not simply product cylinders).*

Proof. It will be enough to show that general cylinders are measurable in the product σ-algebra.

Recall the definition of a cylinder

$$C = \{x \in \prod X_i : (\pi_1(x), \ldots, \pi_n(x)) \in B\}$$

where $B \subseteq X_1 \otimes \cdots \otimes X_n$ (here I took the indices to be $1, \ldots, n$ without loss of generality).

Define a map $F : \prod X_i \to \prod_1^n$ ∎