UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 2

Problem 2.1. Let $(S, \mathcal{S}, \mu)$ be a measure space, and suppose that $f \in \mathcal{L}^1$. Show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{S}$ and $\mu(A) < \delta$, then $\left|\int_A f\, d\mu\right| < \varepsilon$.

Solution: First of all, it is enough to prove the claim for non-negative functions. Indeed, the fact that

$$\left|\int_A f\, d\mu\right| \le \int_A |f|\, d\mu,$$

implies that if the result holds for $|f|$, it also holds for f.

Suppose, to the contrary, that there is a non-negative function $f \in \mathcal{L}^1$, a constant $\varepsilon > 0$ and a sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets in $\mathcal{S}$ such that $\mu(A_n) \to 0$ and $\int_{A_n} f\, d\mu \ge \varepsilon$. By passing to a subsequence (if necessary), we may suppose that, additionally, $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$. With $B_n = \cup_{k\ge n} A_n$, we have $\mu(B_n) \le \sum_{k\ge n} \mu(A_k) \to 0$ as well as

$$\int_{B_n} f\, d\mu \ge \int_{A_n} f\, d\mu \ge \varepsilon. \tag{2.1}$$

On the other hand, with $B = \cap_n B_n = \limsup_n A_n$, we have $\mu(B) = 0$ and $f\mathbf{1}_{B_n} \to f\mathbf{1}_B$. It follows now, by the dominated convergence theorem that $\int f\mathbf{1}_{B_n} \to \int f\mathbf{1}_B = 0$ - a contradiction with (2.1).

Problem 2.2. Let $(S, \mathcal{S}, \mu)$ be a finite measure space. For $f \in \mathcal{L}^0_+$, show that $f \in \mathcal{L}^1$ if and only if

$$\textstyle\sum_{n\in\mathbb{N}} \mu(\{f \ge n\}) < \infty.$$

Solution: Consider the functions $g, h : S \to [0, \infty)$, given by

$$g = \sum_{n=1}^{\infty} n\mathbf{1}_{\{n\le f<n+1\}} \text{ and } h = \sum_{n=0}^{\infty} (n+1)\mathbf{1}_{\{n\le f<n+1\}}.$$

We clearly have $g(x) \le f(x) \le h(x)$, for all $x \in S$, and so $\int g\, d\mu \le \int f\, d\mu \le \int h\, d\mu$. The monotone convergence theorem implies that

$$\int g\, d\mu = \sum_{n=1}^{\infty} n\mu\Big(\{n \le f < n+1\}\Big) \text{ and } \int h\, d\mu = \sum_{n=0}^{\infty} (n+1)\mu\Big(\{n \le f < n+1\}\Big).$$

On the other hand, by the simple identities

$$\sum_{n=0}^{\infty} \mathbf{1}_{\{f\ge n\}} = \sum_{n=0}^{\infty} (n+1)\mathbf{1}_{\{n\le f<n+1\}} \text{ and } \sum_{n=1}^{\infty} \mathbf{1}_{\{f\ge n\}} = \sum_{n=1}^{\infty} n\mathbf{1}_{\{n\le f<n+1\}}$$

and Corollary **??** in the notes, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \mu\Big(\{f \ge n\}\Big) &= \sum_{n=1}^{\infty} \int \mathbf{1}_{\{f\ge n\}}\, d\mu = \int \sum_{n=1}^{\infty} \mathbf{1}_{\{f\ge n\}}\, d\mu = \int \sum_{n=1}^{\infty} n\mathbf{1}_{\{n\le f<n+1\}}\, d\mu \\ &= \sum_{n=1}^{\infty} n\mu\Big(\{n \le f < n+1\}\Big) = \int g\, d\mu,\end{aligned}$$

as well as

$$\begin{aligned}\sum_{n=0}^{\infty} \mu\Big(\{f \ge n\}\Big) &= \sum_{n=0}^{\infty} \int \mathbf{1}_{\{f\ge n\}}\, d\mu = \int \sum_{n=0}^{\infty} \mathbf{1}_{\{f\ge n\}}\, d\mu = \int \sum_{n=0}^{\infty} (n+1)\mathbf{1}_{\{n\le f<n+1\}}\, d\mu \\ &= \sum_{n=0}^{\infty} (n+1)\mu\Big(\{n \le f < n+1\}\Big) = \int h\, d\mu,\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} \mu\Big(\{f \ge n\}\Big) \le \int f\, d\mu \le \sum_{n=1}^{\infty} \mu\Big(\{f \ge n\}\Big),$$

and the claim follows; indeed, since $\mu(f \ge 0) < \infty$, the series above either both converge or both diverge.

Problem 2.3. Let $(S, \mathcal{S}, \mu)$ be a measure space, and suppose $f \in \mathcal{L}^1_+$ is such that $\int f \, d\mu = c > 0$. Show that for each $\alpha > 0$ the limit

$$\lim_n \int n \log\left(1 + (f/n)^\alpha\right) d\mu$$

exists in $[0, \infty]$ and compute its value. *Hint:* Prove and use the inequality $\log(1 + x^\alpha) \leq \alpha x$, valid for $x \geq 0$ and $\alpha \geq 1$.

Solution: Without loss of generality, we may suppose that $f(x) < \infty$, for all $x \in S$ (why?). Define $f_n(x) = n \log\left(1 + (f/n)^\alpha\right)$, for $x \in S$. Using the fact that

$$y \log(1 + \tfrac{r}{y}) \to r \text{ as } y \to \infty, \text{ for } r \geq 0,$$

we easily get that (here $y = n^\alpha$)

$$\lim_n f_n(x) = \lim_{y \to \infty} y^{1/\alpha - 1}\left(y \log(1 + \tfrac{f^\alpha}{y})\right) = \begin{cases} 0, & \alpha > 1 \\ f, & \alpha = 1 \\ \infty, & 0 < \alpha < 1 \end{cases}$$

When $\alpha < 1$, we can use Fatou's lemma (and the fact the $\mu(S) > 0$ - where is it used exactly?) to get

$$\liminf \int f_n \, d\mu \geq \int \lim_n f_n \, d\mu \geq \int \infty \, d\mu = \infty.$$

Therefore $\int f_n \, d\mu \to \infty$.

When $\alpha \geq 1$, we define the function $h(x) = \alpha x - \log(1 + x^\alpha)$ and compute its derivative

$$h'(x) = \alpha - \tfrac{\alpha x^{\alpha-1}}{1+x^\alpha} = \tfrac{\alpha}{1+x^\alpha}\left(1 + x^\alpha - x^{\alpha-1}\right),$$

and deduce that $h'(x) \geq 0$, for all $x > 0$. It follows that h is a non-decreasing function and so $h(x) \geq h(0) = 0$, i.e.

$$\alpha x \geq \log(1 + x^\alpha), \text{ for } x \geq 0.$$

It follows that $f_n \leq \alpha f$, for all $n \in \mathbb{N}$, and so, the dominated convergence theorem can be applied:

$$\lim_n \int f_n \, d\mu = \int \lim_n f_n \, d\mu = \begin{cases} 0, & \alpha > 1 \\ c, & \alpha = 1. \end{cases}$$

Problem 2.4. A sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{L}^0$ is said to **converge in measure** toward $f \in \mathcal{L}^0$ if

$$\forall \varepsilon > 0, \; \mu\Big(\{ |f_n - f| \geq \varepsilon \}\Big) \to 0 \text{ as } n \to \infty.$$

Assume that $\mu(S) < \infty$ (parts marked by (†) are true without this assumption).

1. Show that the mapping
$$d(f, g) = \int \tfrac{|f-g|}{1+|f-g|} \, d\mu, \; f, g \in \mathcal{L}^0,$$
defines a pseudo metric on $\mathcal{L}^0$ and that convergence in d is equivalent to convergence in measure.
2. Show that $f_n \to f$, a.e., implies that $f_n \to f$ in measure. Give an example which shows that the assumption $\mu(S) < \infty$ is necessary.
3. Give an example of a sequence which converges in measure, but not a.e.
4. (†) For $f \in \mathcal{L}^0$ and a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{L}^0$, show that $f_n \to f$, a.s., if the convergence in measure "happens fast", i.e., if
$$\textstyle\sum_{n \in \mathbb{N}} \mu\Big(\{ |f_n - f| \geq \varepsilon \}\Big) < \infty, \text{ for all } \varepsilon > 0.$$
5. (†) Show that each sequence convergent in measure has a subsequence which converges a.e.
6. (†) Show that each sequence convergent in $\mathcal{L}^p$, for some $p \in [1, \infty)$, converges in measure.
7. For $p \in [1, \infty)$, find an example of a sequence which converges in measure, but not in $\mathcal{L}^p$.

8. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^0$ with the property that any of its subsequences admits a (further) subsequence which converges a.e. to $f \in \mathcal{L}^0$. Show that $f_n \to f$ in measure.

9. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, and let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences in $\mathcal{L}^0$. If $f, g \in \mathcal{L}^0$ are such that $f_n \to f$ and $g_n \to g$ in measure, then

$$\Phi(f_n, g_n) \to \Phi(f, g) \text{ in measure.}$$

Note: $\Phi = +$ or $\Phi = \times$ are particularly useful.

Solution:

1. The only non-trivial property of a pseudo metric that needs to be shown is the triangle inequality, and the follows from the fact (to prove it, simply expand everything) that

$$\tfrac{a}{1+a} + \tfrac{b}{1+b} \geq \tfrac{c}{1+c}, \text{ for } a, b, c \geq 0,\ a + b \geq c.$$

For $\{f_n\}_{n\in\mathbb{N}}, f \in \mathcal{L}^0$, $\varepsilon > 0$, we have

$$\tfrac{\varepsilon}{1+\varepsilon}\mathbf{1}_{\{|f_n-f|\geq\varepsilon\}} \leq \tfrac{|f_n-f|}{1+|f_n-f|} \leq \mathbf{1}_{\{|f_n-f|\geq\varepsilon\}} + \varepsilon\mathbf{1}_{\{|f_n-f|<\varepsilon\}},$$

and so

$$\tfrac{\varepsilon}{1+\varepsilon}\mu(\{|f_n - f| \geq \varepsilon\}) \leq d(f_n, f) \leq \mu(\{|f_n - f| \geq \varepsilon\}) + \varepsilon\mu(S).$$

It follows now easily that $\mu(\{|f_n - f| \geq \varepsilon\}) \to 0$ for each $\varepsilon > 0$, if and only $d(f_n, f) \to 0$.

2. If $f_n \to f$, a.e., then $\frac{|f_n-f|}{1+|f_n-f|} \to 0$, a.e. Moreover,

$$\tfrac{|f_n-f|}{1+|f_n-f|} \leq 1,$$

and the constant 1 is in $\mathcal{L}^1$ (remember $\mu(S) < \infty$), so the dominated convergence theorem implies that $d(f_n, f) \to 0$. By (1), $f_n \to f$, in measure.

For a counterexample, take $(S, \mathcal{S}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, and define $f_n = \mathbf{1}_{[n,\infty)}$. Then $f_n(x) \to 0$, for all $x \in \mathbb{R}$. On the other hand $\lambda(\{|f_n| \geq \varepsilon\}) = \lambda([n, \infty)) = \infty$, for all $1 \geq \varepsilon > 0$ and all $n \in \mathbb{N}$.

3. Take $(S, \mathcal{S}, \mu) = ([0, 1], \mathcal{B}([0, 1]), \mu)$, and define the sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}^0_+$ by

$$\begin{aligned}
&f_1 = 1,\\
&f_2 = \mathbf{1}_{[0,1/2]}, f_3 = \mathbf{1}_{(1/2,1]},\\
&f_4 = \mathbf{1}_{[0,1/4]}, f_5 = \mathbf{1}_{(1/4,1/2]}, f_6 = \mathbf{1}_{(1/2,3/4]}, f_7 = \mathbf{1}_{(3/4,1]},\\
&f_8 = \mathbf{1}_{[0,1/8]}, \text{etc.}
\end{aligned}$$

For $1 \geq \varepsilon > 0$ and $N \in \mathbb{N}$, it suffices to take $n \geq 2^N$ to conclude that $\lambda(\{|f_n| \geq \varepsilon\}) \leq 2^{-N}$. Therefore, $f_n \to 0$ in measure.

On the other hand, it is clear that for each $x \in [0, 1]$ and each $N \in \mathbb{N}$, there exists m, n between 2^N and $2^{N+1} - 1$ so that $f_m(x) = 0$ and $f_n(x) = 1$ (note that n is unique, but m is not for $N > 1$). Consequently $\limsup_n f_n(x) = 1$ and $\liminf_n f_n(x) = 0$ and so $\{f_n(x)\}_{n\in\mathbb{N}}$ diverges for each $x \in [0, 1]$.

4. For $k > 0$, and $n \in \mathbb{N}$, define

$$A^k_n = \{x \in S : |f_n(x) - f(x)| \geq 1/k\}.$$

Then $\sum_{n\in\mathbb{N}} \mu(A^k_n) < \infty$, and the Borel-Cantelli lemma implies that

$$\mu(N^k) = 0, \text{ where } N^k = \{A^k_n \text{ i.o.}\}.$$

For $x \in (N^k)^c$, we have

$$|f_n(x) - f(x)| < \tfrac{1}{k} \text{ for large enough } n \in \mathbb{N},$$

and so

$$\limsup_n |f_n - f| \leq 1/k \text{ on } (N^k)^c.$$

Consequently, $\lim f_n(x) = f(x)$, for all x in the complement of the null set $N = \cup_k N^k$.

5. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence convergent in measure to $f \in \mathcal{L}^0$. For $n, k \in \mathbb{N}$, define

$$a_{n,k} = \mu(\{|f_n - f| \geq 1/k\}),$$

so that $a_{n,k} \to 0$, as $n \to \infty$, for all $k \in \mathbb{N}$. Therefore, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that

$$n_k \geq n_{k-1} \text{ for } k > 1 \text{ and } a_{n,k} \leq 2^{-k}, \text{ for all } n \geq n_k.$$

Consider the subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and pick $\varepsilon > 0$ and $k_0 > 1/\varepsilon$. For $k \geq k_0$ we have $n_k \geq n_{k_0}$ and so

$$\mu(|f_{n_k} - f| \geq \varepsilon) \leq \mu(|f_{n_k} - f| \geq 1/k_0) \leq 2^{-k},$$

so that

$$\sum_{k\in\mathbb{N}} \mu(|f_{n_k} - f| \geq \varepsilon) < \infty, \text{ for each } \varepsilon > 0.$$

By (4) above, $f_{n_k} \to f$, a.e.

6. Assume that $f_n \to f$ in $\mathcal{L}^p$. By Markov's inequality

$$\mu(\{|f_n - f| \geq \varepsilon\}) = \mu(\{|f_n - f|^p \geq \varepsilon^p\}) \leq \varepsilon^{-p} \int |f_n - f|^p \, d\mu$$
$$= \varepsilon^{-p} ||f_n - f||^p_{\mathcal{L}^p} \to 0.$$

7. Take $(S, \mathcal{S}, \mu) = ([0,1], \mathcal{B}([0,1]), \lambda)$ and define $f_n = n\mathbf{1}_{[0,1/n]}$. Then for $\varepsilon > 0$, we have

$$\mu(\{|f_n| \geq \varepsilon\}) \leq \mu(\{|f_n| \geq \min(1,\varepsilon)\}) = \lambda([0,1/n]) = 1/n \to 0.$$

On the other hand

$$||f_n||_{\mathcal{L}^p} = n^p \tfrac{1}{n} \geq 1, \text{ for all } n \in \mathbb{N}.$$

8. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ converges in measure to $f \in \mathcal{L}^0$. Than any subsequence of $\{f_n\}_{n\in\mathbb{N}}$ converges in measure, too, and, by (5), it admits a (further) subsequence which converges to f, a.e.

Conversely, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^0$ with the property that each one of its subsequences admits a further subsequence which converges to $f \in \mathcal{L}^0$, a.e. Suppose that $f_n \not\to f$ in measure. That means that there exists $\varepsilon > 0$ such that

$$\mu(\{|f_n - f| \geq \varepsilon\}) \not\to 0,$$

i.e., that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ and $\delta > 0$ such that

$$\mu(\{|f_{n_k} - f| \geq \varepsilon\}) \geq \delta \text{ for all } k \in \mathbb{N}. \tag{2.2}$$

By assumption, this particular subsequence has a further subsequence $\{f_{n_{k_l}}\}_{l\in\mathbb{N}}$ which converges to f, a.e. Since a.e. convergence implies convergence in measure, we have

$$\mu(\{|f_{n_{k_l}} - f| \geq \varepsilon\}) \to 0, \text{ as } l \to \infty. \tag{2.3}$$

The equations (2.2) and (2.3) are in contradiction.

9. For a sequence $\{n_k\}_{k\in\mathbb{N}}$, let $\{n_{k_l}\}_{k\in\mathbb{N}}$ be such that $f_{n_{k_l}}$ and $g_{n_{k_l}}$ converge towards f and g (respectively), a.e. Then

$$\Phi(f_{n_{k_l}}, g_{n_{k_l}}) \to \Phi(f,g), \text{ a.e.},$$

because an application of a continuous function clearly preserves a.e. convergence. Therefore, for an arbitrary subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$, we have constructed a further subsequence of $\{\Phi(f_{n_k}, g_{n_k})\}_{k\in\mathbb{N}}$ which converges a.e. to $\Phi(f,g)$. By (8), $\Phi(f_n, g_n) \to \Phi(f,g)$ in probability.