UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 2

Problem 2.1. Let $(S, \mathcal{S}, \mu)$ be a *finite* measure space. We have shown that

$$\mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n),$$

for any sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}$. Give an example of a measure space $(S, \mathcal{S}, \mu)$ and on it a (single) sequence $\{A_n\}_{n\in\mathbb{N}}$ for which all inequalities above are strict.

Solution: Let $S = \{1, 2\}$, and let the measure μ on $\mathcal{S} = 2^S$ be given by $\mu(\{1\}) = 1$, $\mu(\{2\}) = 2$. Define

$$A_{2k-1} = \{1\}, A_{2k} = \{2\}, \text{ for } k \in \mathbb{N}.$$

Then $\liminf_n A_n = \emptyset$ and $\limsup_n A_n = S$ so that

$$\mu(\liminf A_n) = 0 \text{ and } \mu(\limsup_n A_n) = 3.$$

The sequence $\{\mu(A_n)\}_{n\in\mathbb{N}}$ alternates between 1 and 2, so that

$$\liminf_n \mu(A_n) = 1 \text{ and } \limsup_n \mu(A_n) = 2.$$

Problem 2.2. Let $(S, \mathcal{S}, \mu)$ be a measure space and let $f, g \in \mathcal{L}^0(S, \mathcal{S}, \mu)$ satisfy $\mu\big(\{x \in S : f(x) < g(x)\}\big) > 0$. Prove or construct a counterexample for the following statement:

"There exist constants $a, b \in \mathbb{R}$ such that $\mu\big(\{x \in S : f(x) \leq a < b \leq g(x)\}\big) > 0$."

Solution: The statement is true. Let $A = \{x \in S : f(x) < g(x)\}$. Since for each $x \in S$ with $f(x) < g(x)$ we can find two rational numbers $a < b$ with $f(x) \leq a < b \leq g(x)$, we have

$$A = \bigcup_{a,b \text{ rational}} \{x \in S : f(x) \leq a < b \leq g(x)\}.$$

The set of all pairs of rational numbers is countable, so by the subadditivity of measure we have

$$0 < \mu(A) \leq \sum_{a,b \text{ rational}} \mu(\{x \in S : f(x) \leq a < b \leq g(x)\}).$$

It follows that we must have $\mu(\{x \in S : f(x) \leq a < b \leq g(x)\}) > 0$ for at least one (rational) pair $a < b$.

Problem 2.3. A measure space $(S, \mathcal{S}, \mu)$ is called **complete** if all subsets of null sets are themselves in $\mathcal{S}$. For a (possibly incomplete) measure space $(S, \mathcal{S}, \mu)$ we define the **completion** $(S, \mathcal{S}^*, \mu^*)$ in the following way:

$$\mathcal{S}^* = \{A \cup N^* : A \in \mathcal{S} \text{ and } N^* \subseteq N \text{ for some } N \in \mathcal{S} \text{ with } \mu(N) = 0\}.$$

For $B \in \mathcal{S}^*$ with representation $B = A \cup N^*$ we set $\mu^*(B) = \mu(A)$.

1. Show that $\mathcal{S}^*$ is a σ-algebra.
2. Show that the definition $\mu^*(B) = \mu(A)$ above does not depend on the choice of the decomposition $B = A \cup N^*$, i.e., that $\mu(\hat{A}) = \mu(A)$ if $B = \hat{A} \cup \hat{N}^*$ is another decomposition of B into a set $\hat{A}$ in $\mathcal{S}$ and a subset $\hat{N}$ of a null set in $\mathcal{S}$.
3. Show that μ^* is a measure on $(S, \mathcal{S}^*)$ and that $(S, \mathcal{S}^*, \mu^*)$ is a complete measure space with the property that $\mu^*(A) = \mu(A)$, for $A \in \mathcal{S}$.

Solution:

1. We prove only the complement axiom here (the others are easier). Let $B = A \cup N^* \in \mathcal{F}$, where $A \in \mathcal{S}$ and $N^* \subseteq N$, and N is a null set in $\mathcal{S}$. Then

$$B^c = A^c \cap (N^*)^c = (A^c \cap N^c) \cup (A^c \cap (N^*)^c \cap N) \in \mathcal{S}^*,$$

because $A^c \cap N^c \in \mathcal{F}$ and $(A^c \cap (N^*)^c \cap N) \subseteq N$ is a subset of a null set.

2. Suppose $C = A \cup N^* = B \cup M^*$ with $A, B \in \mathcal{F}$ and N^*, M^* subsets of null sets N and M in $\mathcal{S}$. Then

$$A \subseteq A \cup N^* = C = B \cup M^* \subseteq B \cup M,$$

and so $\mu(A) \leq \mu(B \cup M) \leq \mu(B) + \mu(M) = \mu(B)$. Similarly, $\mu(B) \leq \mu(A)$, and so $\mu(A) = \mu(B)$.

3. Let $\{B_n\}_{n\in\mathbb{N}}$ be a pairwise disjoint sequence in $\mathcal{S}^*$. By definition, each B_n has a representation $B_n = A_n \cup N_n^*$, for some $A_n \in \mathcal{S}$ and $N_n^* \subseteq N_n \in \mathcal{S}$ with $\mu(N_n) = 0$. Also $\mu^*(B_n) = \mu(A_n)$.
Since $A_n \subseteq B_n$, the sequence $\{A_n\}_{n\in\mathbb{N}}$ is pairwise disjoint in $\mathcal{S}$, and, with $A = \cup_{n\in\mathbb{N}} A_n$, we have

$$\mu(A) = \sum_{n\in\mathbb{N}} \mu(A_n) = \sum_{n\in\mathbb{N}} \mu^*(B_n).$$

On the other hand, the set $B = \cup_{n\in\mathbb{N}} B_n$ can be written as

$$B = A \cup N^*, \text{ where } N^* = \cup_{n\in\mathbb{N}} N_n^*. \tag{2.1}$$

Since $A \in \mathcal{S}$ and N^* is a subset of the null set $N = \cup_{n\in\mathbb{N}} N_n$ in $\mathcal{S}$, we have $\mu^*(B) = \mu(A)$ and the statement follows from (2.1).

Problem 2.4. The measure space $(S, \mathcal{S}, \mu)$, where (S, d) is a metric space and $\mathcal{S}$ is a σ-algebra on S which contains the Borel σ-algebra $\mathcal{B}(d)$ on S is called **regular** if for each $A \in \mathcal{S}$ and each $\varepsilon > 0$ there exist a closed set C and an open set O such that $C \subseteq A \subseteq O$ and $\mu(O \setminus C) < \varepsilon$.

1. Suppose that $(S, \mathcal{S}, \mu)$ is a regular measure space, and that the measure space $(S, \mathcal{B}(d), \mu|_{\mathcal{B}(d)})$ is obtained from $(S, \mathcal{S}, \mu)$ by restricting the measure μ onto the σ-algebra of Borel sets. Show that $\mathcal{S} \subseteq \mathcal{B}(d)^*$, where $\big(S, \mathcal{B}(d)^*, (\mu|_{\mathcal{B}(d)})^*\big)$ is the completion of $\big(S, \mathcal{B}(d), \mu|_{\mathcal{B}(d)}\big)$
2. Suppose that (S, d) is a metric space and that μ is a finite measure on $\mathcal{B}(d)$. Show that $(S, \mathcal{B}(d), \mu)$ is a regular measure space. *Hint:* Consider a collection $\mathcal{A}$ of subsets A of S such that for each $\varepsilon > 0$ there exists a closed set C and an open set O with $C \subseteq A \subseteq O$ and $\mu(O \setminus C) < \varepsilon$. Argue that $\mathcal{A}$ is a σ-algebra. Then show that each closed set can be written as an intersection of open sets; use (but prove, first) the fact that the map

$$x \mapsto d(x, C) = \inf\{d(x, y) : y \in C\},$$

is continuous on S for any nonempty $C \subseteq S$.

3. Show that $(S, \mathcal{B}(d), \mu)$ is regular if μ is not necessarily finite, but has the property that $\mu(A) < \infty$ whenever $A \in \mathcal{B}(d)$ is bounded, i.e., when $\sup\{d(x, y) : x, y \in A\} < \infty$. *Hint:* Pick a point $x_0 \in S$ and, for $n \in \mathbb{N}$, define the family $\{R_n\}_{n\in\mathbb{N}}$ of subsets of S as follows:

$$R_1 = \{x \in S : d(x, x_0) < 2\}, \text{ and}$$
$$R_n = \{x \in S : n-1 < d(x, x_0) < n+1\}, \text{ for } n > 1,$$

as well as a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of set functions on $\mathcal{B}(d)$, given by $\mu_n(A) = \mu(A \cap R_n)$, for $A \in \mathcal{B}(d)$. Under the right circumstances, even countable unions of closed sets are closed.

4. Conclude that the Lebesgue measure on $\big(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\big)$ is regular.

Solution:

1. We need to show that each B in $\mathcal{S}$ can be written in the form $B = A \cup N^*$, where $A \in \mathcal{B}(d)$ and $N^* \subseteq N$, for some $N \in \mathcal{B}(d)$ with $\mu(N) = 0$. By regularity, there exist two sequences $\{C_n\}_{n\in\mathbb{N}}$ and $\{O_n\}_{n\in\mathbb{N}}$ of closed and open sets, respectively, such that

$$C_n \subseteq B \subseteq O_n \text{ and } \mu(O_n \setminus C_n) < 2^{-n}, \ \forall\, n \in \mathbb{N}.$$

For $n \in \mathbb{N}$, define

$$\hat{C}_n = \cup_{k\geq n} C_k \text{ and } \hat{C} = \cap_n \hat{C}_n = \limsup_n C_n,$$

and, similarly,

$$\hat{O}_n = \cap_{k\geq n} O_k \text{ and } \hat{O} = \cup_n \hat{O}_n = \limsup_n O_n,$$

so that, clearly, $\{\hat{C}_n\}_{n\in\mathbb{N}}, \{\hat{O}_n\}_{n\in\mathbb{N}}, \hat{C}, \hat{O} \in \mathcal{B}(d)$.
Moreover, since $\hat{O}_n \supseteq \hat{O}$ and $\hat{C}_n \subseteq \hat{C}$, for all $n \in \mathbb{N}$, we have $\hat{C} \subseteq B \subseteq \hat{O}$ and

$$\begin{aligned}\mu(\hat{O}\setminus\hat{C}) &\leq \mu(\hat{O}_n\setminus\hat{C}_n) = \mu\big(\cap_{k\geq n} O_k \cap (\cup_{k\geq n} C_k)^c\big) = \mu\big(\cap_{k\geq n}(O_k\cap C_k^c)\big)\\ &\leq \sum_{k\geq n}\mu(O_k\setminus C_k) \leq 2^{-n+1}, \text{ for all } n\in\mathbb{N}.\end{aligned}$$

Consequently, $\mu(\hat{O}\setminus\hat{C}) = 0$, i.e., $N = \hat{O}\setminus\hat{C}$ is a null set in $\mathcal{B}(d)$. The required decomposition of B is given by $B = \hat{C}\cup N^*$. where $N^* = B\setminus\hat{C}\subseteq N$.

2. Let $\mathcal{A}$ be the collection of all subsets A of S with the property that for each $\varepsilon > 0$ there exists a closed set C and an open set O with $C \subseteq A \subseteq O$ and $\mu(O\setminus C) < \varepsilon$. It is straightforward to check that the set $\mathcal{A}$ is an algebra (simply use the fact that finite unions and intersections of open (closed) sets are open (closed)). To show σ-additivity, we take a pairwise disjoint sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ with $A = \cup_n A_n$. For $\varepsilon > 0$, let $\{C_n\}_{n\in\mathbb{N}}$ and $\{O_n\}_{n\in\mathbb{N}}$ be sequences of open and closed sets, respectively, such that $C_n \subseteq A_n \subseteq O_n$ and $\mu(O_n\setminus C_n) < \frac{\varepsilon}{3}2^{-n}$. Let $N\in\mathbb{N}$ be such that $\sum_{n>N}\mu(A_n) < \varepsilon/3$ - such N exists thanks to the fact that $\{A_n\}_{n\in\mathbb{N}}$ is pairwise disjoint and the assumption that $\mu(A)\leq\mu(S) < \infty$. Define

$$C = C_1\cup\cdots\cup C_N, \text{ and } O = \cup_{n\in\mathbb{N}} O_n,$$

so that C is a closed set and O is open and $C\subseteq A\subseteq O$. Furthermore, we have

$$\begin{aligned}\mu(O\setminus C) &= \mu\Big(\big(\cup_{n\in\mathbb{N}} O_n\big)\cap C_1^c\cap\cdots\cap C_N^c\Big) = \mu\Big(\cup_{n\in\mathbb{N}}\big(O_n\cap C_1^c\cap\cdots\cap C_N^c\big)\Big)\\ &\leq \sum_{n\in\mathbb{N}}\mu(O_n\cap(C_1^c\cap\cdots\cap C_N^c)) \leq \sum_{n=1}^N \mu(O_n\setminus C_n) + \sum_{n=N+1}^\infty \mu(O_n)\\ &\leq \sum_{n=1}^\infty \tfrac{\varepsilon}{3}2^{-n} + \sum_{n=N+1}^\infty\Big(\mu(A_n)+\mu(O_n\setminus C_n)\Big) < \tfrac{\varepsilon}{3}+\tfrac{\varepsilon}{3}+\sum_{n=N+1}^\infty \tfrac{\varepsilon}{3}2^{-n} < \varepsilon.\end{aligned}$$

Therefore $\mathcal{A}$ is a σ-algebra, so it will be enough to show that $\mathcal{A}$ contains all closed sets. For a closed set C and $n\in\mathbb{N}$, define

$$O_n = \{x\in S : d(x,C) < \tfrac{1}{n}\},$$

where $d(x,C) = \inf\{d(x,y) : y\in C\}$. For $x_1, x_2\in S$ and $y\in C$, we have

$$d(x_1,C) - d(x_1,x_2) \leq d(x_1,y) - d(x_2,x_1) \leq d(x_2,y),$$

and so

$$d(x_1,C) - d(x_1,x_2) \leq \inf_{y\in C} d(x_2,y) = d(x_2,C).$$

Similarly,

$$d(x_2,C) - d(x_1,x_2) \leq d(x_1,C),$$

so that

$$|d(x_1,C) - d(x_2,C)| \leq d(x_1,x_2), \text{ for all } x_1,x_2\in S,$$

which implies that the function $x\mapsto d(x,C)$, form S to $[0,\infty)$ is Lipschitz and, therefore, continuous. This, in turn, implies that O_n is an open set. Moreover, for $x\in\cap_n O_n$, we have $d(x,C) = 0$, i.e., there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C with $d(x_n,x)\to 0$. By closedness of C, we have $x = \lim_n x_n \in C$, and so

$$C = \cap_n O_n.$$

Since $\mu(S) < \infty$, the continuity of measure implies that $\mu(C) = \lim_{n\in\mathbb{N}}\mu(O_n)$, so, for large-enough $n\in\mathbb{N}$, we have $\mu(O_n\setminus C) < \varepsilon$.

3. Pick a point $x_0\in S$ and, for $n\in\mathbb{N}$, define the family $\{R_n\}_{n\in\mathbb{N}}$ of subsets of S as follows:

$$\begin{aligned}R_1 &= \{x\in S : d(x,x_0) < 2\}, \text{ and}\\ R_n &= \{x\in S : n-1 < d(x,x_0) < n+1\}, \text{ for } n > 1.\end{aligned}$$

All R_1 are open, and so the set function $\mu_n : \mathcal{B}(d)\to[0,\infty)$, given by $\mu_n(A) = \mu(A\cap R_n)$, for $A\in\mathcal{B}(d)$, is well-defined. The assumption implies that $\mu(R_n) < \infty$ and it is easy to check that μ_n is a finite - and, thus, regular - *measure*.
Given $B\in\mathcal{B}(d)$, and $\varepsilon > 0$, $n\in\mathbb{N}$ we can find an open set $O_{\varepsilon,n}$ and a closed set $C_{\varepsilon,n}$ so that $C_{\varepsilon,n}\subseteq B\cap R_n\subseteq O_{\varepsilon,n}$ and $\mu_n(O_{\varepsilon,n}\setminus C_{\varepsilon,n}) < \varepsilon 2^{-n}$. Given that R^n is open and $\mu_n(R_n^c) = 0$, we can (and do) replace $O_{\varepsilon,n}$ by $O_{\varepsilon,n}\cap R_n$ without changing any conclusions. We do this so that we can assume that $C_{\varepsilon,n}, O_{\varepsilon,n}\subseteq R_n$, for all $n\in\mathbb{N}$.
Set $O_\varepsilon = \cup_n O_{n,\varepsilon}$ and $C_\varepsilon = \cup_n C_n$, so that O_ε is clearly open and $C_\varepsilon\subseteq B\subseteq C_\varepsilon$. Next, we show that that C_ε is closed. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in C_ε such that $x_n\to x$, for some $x\in S$. Then $\{x_n\}_{n\in\mathbb{N}}$ is necessarily bounded, and since $C_{\varepsilon,n}\subseteq R_n$, there must exist $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$, $x_n\in C_{\varepsilon,1}\cup\cdots\cup C_{\varepsilon,N}$, which is a closed set. Therefore $x\in C_{\varepsilon,1}\cup\cdots\cup C_{\varepsilon,N}\subseteq C_\varepsilon$ (because $d(x,x_0) > n-1$, for $x\in C_{\varepsilon,n}$).
Since $O_{\varepsilon,n}\subseteq R_n$, we have $\mu(O_{\varepsilon,n}\setminus C_{\varepsilon,n}) = \mu_n(O_{\varepsilon,n}\setminus C_{\varepsilon,n})\leq\varepsilon 2^{-n}$, and so

$$\mu(O_\varepsilon\setminus C_\varepsilon) = \mu\Big(\cup_{n\in\mathbb{N}} O_{\varepsilon,n}\setminus C_\varepsilon\Big) \leq \sum_{n\in\mathbb{N}}\mu(O_{\varepsilon,n}\setminus C_{\varepsilon,n}) \leq \varepsilon.$$