UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 1

Problem 1.1. A **partition** of a set S is a family $\mathcal{P}$ of non-empty subsets of S with the property that each $\omega \in S$ belongs to exactly one $A \in \mathcal{P}$.

1. Show that the number of different algebras on a finite set S is equal to the number of different partitions of S. *Note:* This number for $S_n = \{1, 2, \dots, n\}$ is called the n^{th} **Bell number** B_n, and no nice closed-form expression for it is known. See 5. below, though.
2. How many algebras are there on the set $S = \{1, 2, 3\}$?
3. Does there exist an algebra with 754 elements?
4. For $N \in \mathbb{N}$, let a_n be the number of different algebras on the set $\{1, 2, \dots, n\}$. Show that $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, and that the following recursion holds (where $a_0 = 1$ by definition),
$$a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k.$$
5. Show that the exponential generating function for the sequence $\{a_n\}_{n\in\mathbb{N}}$ is $f(x) = e^{e^x-1}$, i.e., that
$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = e^{e^x-1} \text{ or, equivalently, } a_n = \left(\tfrac{d^n}{dx^n} e^{e^x-1} \right) \Big|_{x=0}.$$

Solution:

1. For a partition $\mathcal{P} = \Big\{A_1, \dots, A_m\Big\}$ of S, consider the set $\mathcal{A}(\mathcal{P})$ which consists of the empty set and all sets of the form $A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}$, where $1 \leq i_1 < i_2 < \dots i_k \leq m$. It is clear that $\mathcal{A}(\mathcal{P})$ is an algebra. Conversely, for an algebra $\mathcal{A}$, let the family $\mathcal{P}(\mathcal{A})$ of subsets of S be given by
$$\mathcal{P}(\mathcal{A}) = \left\{ \bigcap_{A \ni x} A \, : \, x \in S \right\}.$$
In words, the partition element containing $x \in S$ will be the intersection of all algebra elements which contain x. It is easy to show that $\mathcal{P}(\mathcal{A})$ is a partition. Moreover, the map $\mathcal{P} \mapsto \mathcal{P}(\mathcal{A})$ is one-to-one and onto. Therefore, the number of partitions is equal to the number of algebras.
2. The number of algebras is equal to the number of partitions of $\{1, 2, 3\}$, and all partitions of $\{1, 2, 3\}$ are
$$\Big\{\{1\}, \{2\}, \{3\}\Big\}, \; \Big\{\{1, 2\}, \{3\}\Big\}, \; \Big\{\{1\}, \{2, 3\}\Big\}, \; \Big\{\{2\}, \{1, 3\}\Big\}, \text{ and } \Big\{\{1, 2, 3\}\Big\}.$$
Therefore, there are 5.
3. Let $\mathcal{A}$ be an algebra on S, and let $\mathcal{P} = \mathcal{P}(\mathcal{A})$ be the corresponding partition, and let $m \in \mathbb{N}$ be the number of elements of $\mathcal{P}$. Then, by (1) above, each element of the algebra $\mathcal{A}$ corresponds to a union of some sub-collection of $\mathcal{P}$. Equivalently, if we number the elements of the partition $\mathcal{P}$ as $\{1, 2, \dots, m\}$, then each element of the algebra corresponds to a (possibly empty) subset of $\{1, 2, \dots, m\}$. There are 2^m subsets of $\{1, 2, \dots, m\}$, therefore, the algebra $\mathcal{A}$ has 2^m elements. Since 754 is not a power of 2, there is no algebra with exactly 754 elements.
4. We use mathematical induction. It is clearly true for $n = 0, 1, 2$ (it can be checked directly). The induction hypothesis is that it holds for all m-element sets, for $m \leq n$. Let $P(n+1)$ denote the collection of all partitions of the set $\{1, 2, \dots, n+1\}$, and let $P(n+1; k)$ denote the set of all partitions of $\{1, 2, \dots, n+1\}$ with the property that 1 is in a partition member with exactly k elements. Clearly, we have
$$P(n+1) = \bigcup_{k=1}^{n+1} P(n+1; k), \tag{1.1}$$
where the sets in the union are disjoint. Therefore, $a_{n+1} = \#P(n+1) = \sum_{k=1}^{n+1} \#P(n+1; k)$, where, for a set C, $\#C$ denotes the number of elements of C. It remains to compute $\#P(n; k)$, for $k = 1, \dots, n+1$.

   When $k = 1$, the elements of $P(n+1; 1)$ are of the form $\Big\{\{1\}, A_1, \dots, A_l\Big\}$, where $A_1, \dots, A_l$ form a partition of the set $\{2, 3, \dots, n+1\}$. By the inductive hypothesis, we know that there are a_n $\left(= \binom{n}{0} a_n\right)$ such partitions.

For $k = 2$, the elements of the set $P(n+1;2)$ are of the form $\Big\{\{1,r\}, A_1, \dots, A_l\Big\}$, where $r \in \{2,\dots,n+1\}$ and $A_1,\dots,A_l$ is a partition of the set $\{2,3,\dots,r-1,r+1,\dots,n+1\}$; there are a_{n-1} of those for a fixed r. On the other hand we can choose r from $\{2,3,\dots,n+1\}$ in $\binom{n}{1}$ different ways. Therefore, $\#P(n+1;2) = \binom{n}{1}a_{n-1}$.
Similarly, to compute $\#P(n+1;k)$, we first choose the remaining $k-1$ elements of the partition-member which contains 1, which can be done in $\binom{n}{k-1}$ ways. The rest of the elements can be split into a partition in a_{n+1-k} ways.
Finally, the above discussion and (1.1) above imply that

$$a_{n+1} = \sum_{k=1}^{n+1} \binom{n}{k-1} a_{n+1-k} = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}.$$

5. Remember the Leibnitz formula

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)},$$

where $()^{(n)}$ denotes the n-th derivative. Since for $h(x) = e^{e^x-1}$,

$$h'(x) = h(x)e^x,$$

we have

$$h^{(n+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} h^{(k)}(x)e^x.$$

For $x = 0$, we see that the sequence $\{h^{(n)}(0)\}_{n\in\mathbb{N}}$ satisfies the same recursive equation as the sequence $\{a_n\}_{n\in\mathbb{N}}$. It is straightforward to check that the initial values a_0 and $h^{(0)}$ also agree. Therefore, $a_n = h^{(n)}(0)$.

Problem 1.2. One can obtain the product σ-algebra $\mathcal{S}$ on $\{-1,1\}^{\mathbb{N}}$ as the Borel σ-algebra corresponding to a particular topology which makes $\{-1,1\}^{\mathbb{N}}$ compact. Here is how. Start by defining a mapping $d : \{-1,1\}^{\mathbb{N}} \times \{-1,1\}^{\mathbb{N}} \to [0,\infty)$ by

$$d(\mathbf{s}^1,\mathbf{s}^2) = 2^{-i(\mathbf{s}^1,\mathbf{s}^2)}, \text{ where } i(\mathbf{s}^1,\mathbf{s}^2) = \inf\{i \in \mathbb{N} : s^1_i \neq s^2_i\}, \tag{1.2}$$

for $\mathbf{s}^j = (s^j_1, s^j_2, \dots)$, $j = 1,2$.

1. Show that d is a metric on $\{-1,1\}^{\mathbb{N}}$.
2. Show that $\{-1,1\}^{\mathbb{N}}$ is compact under d. *Hint:* Use the diagonal argument.
3. Show that each cylinder of $\{-1,1\}^{\mathbb{N}}$ is both open and closed under d.
4. Show that each open ball is a cylinder.
5. Show that $\{-1,1\}^{\mathbb{N}}$ is separable, i.e., it admits a countable dense subset.
6. Conclude that $\mathcal{S}$ coincides with the Borel σ-algebra on $\{-1,1\}^{\mathbb{N}}$ under the metric d.

Solution:

1. All the axioms except the triangle inequality are trivial to check. In order to prove the triangle inequality, we pick $\mathbf{s}^1, \mathbf{s}^2$ and $\mathbf{s}^3$ in $\{-1,1\}^{\mathbb{N}}$ and set $i = \min(i(\mathbf{s}^1,\mathbf{s}^2), i(\mathbf{s}^2,\mathbf{s}^3))$. By the definition of i, we have $s^1_k = s^2_k$, $s^2_k = s^3_k$ for $k = 1,2,\dots,i-1$. Therefore, $i(\mathbf{s}^1,\mathbf{s}^3) \geq i$. Consequently

$$d(\mathbf{s}^1,\mathbf{s}^3) \leq \max\Big(d(\mathbf{s}^1,\mathbf{s}^2), d(\mathbf{s}^2,\mathbf{s}^3)\Big) \tag{1.3}$$

which implies the triangle inequality because

$$\max\Big(d(\mathbf{s}^1,\mathbf{s}^2), d(\mathbf{s}^2,\mathbf{s}^3)\Big) \leq d(\mathbf{s}^1,\mathbf{s}^2) + d(\mathbf{s}^2,\mathbf{s}^3).$$

Note: A metric which satisfies the stronger version (1.3) of the triangle inequality is called the **ultra-metric**.

2. Start with a sequence $\{\mathbf{s}^n\}_{n\in\mathbb{N}}$ in $\{-1,1\}^{\mathbb{N}}$, and observe that the sequence of the first coordinates $\{s^n_1\}_{n\in\mathbb{N}}$ takes values in the set $\{-1,1\}$, so it must have a *constant* subsequence, which we denote by $\{s^{n_{1,k}}_1\}_{k\in\mathbb{N}}$. The sequence of second coordinates $\{s^{n_{1,k}}_2\}_{k\in\mathbb{N}}$ considered only on the indices of the first subsequence is also a sequence in $\{-1,1\}$, so it admits a (further) subsequence $\{s^{n_{2,k}}_2\}_{k\in\mathbb{N}}$, which is eventually constant. Continue, extracting subsequences from previous subsequences, and use the diagonal argument, i.e., consider the sequence $(n_{1,1}, n_{2,2}, n_{3,3}, \dots)$. Each coordinate of the sequence $\{\mathbf{s}^{n_{l,l}}\}_{l\in\mathbb{N}}$, which is extracted from $\{\mathbf{s}^n\}_{n\in\mathbb{N}}$ along these indices, has the property that it is constant after a point. Define $\mathbf{s} \in \{-1,1\}^{\mathbb{N}}$ so that s_n is equal to the value that the n-th coordinate that this subsequence stabilizes at. Then, for each $k \in \mathbb{N}$, there exists $L \in \mathbb{N}$ such that the first k coordinates of $\mathbf{s}^{n_{l,l}}$ are equal to those of $\mathbf{s}$ for $l \geq L$, i.e., at a distance of at most 2^{-k} from it.

3. Each cylinder C in $\{-1,1\}^{\mathbb{N}}$ is specified by $n \in \mathbb{N}$ and a subset B of $\{-1,1\}^n$ by

$$C = \Big\{ \mathbf{s} \in \{-1,1\}^{\mathbb{N}} : (s_1, \dots, s_n) \in B \Big\}. \tag{1.4}$$

Since B is necessarily a finite set, each cylinder is a finite union of product cylinders

$$C = \bigcup_{(b_1,\dots,b_n)\in B} C_{1,\dots,n;b_1,\dots,b_n},$$

where

$$C_{n_1,\dots,n_k;b_1,\dots,b_k} = \Big\{ \mathbf{s} \in \{-1,1\}^{\mathbb{N}} : s_{n_1} = b_1, \dots, s_{n_k} = b_k. \Big\}$$

It is therefore enough to show that each product cylinder of the form $C_{1,\dots,n;b_1,\dots,b_n}$ is both open and closed. It is open, because, as you can easily check, each such cylinder is an open ball

$$C_{1,\dots,n;b_1,\dots,b_n} = B(\mathbf{s}_0, 2^{-n}), \text{ where } \mathbf{s}_0 = (b_1, \dots, b_n, 1, 1, \dots).$$

On the other hand, it is closed because its complement is open. Indeed, it is a cylinder of the form (1.4), where $B = \{-1,1\}^n \setminus \{b_1, \dots, b_n\}$, which is, itself, a union of $2^n - 1$ cylinders of the form $C_{1,\dots,n;b'_1,\dots,b'_n}$.

4. Pick $\mathbf{s} \in \{-1,1\}^{\mathbb{N}}$ and $r > 0$ and set $B = B(\mathbf{s}, r)$. Since the distance function takes values in the set $\{2^{-1}, 2^{-2}, \dots\}$, we can assume that $r = 2^{-n}$ for some n. Finally, given that $d(\mathbf{s}, \mathbf{s}') < 2^{-n}$ if and only if $\mathbf{s}'$ and $\mathbf{s}$ agree on the first n coordinates, it is clear that

$$B = C_{1,\dots,n;s_1,\dots,s_n}.$$

5. The required dense set D is given by the collection of all $\mathbf{s} \in \{-1,1\}^{\mathbb{N}}$ such that there exists $N \in \mathbb{N}$ (possibly depending on $\mathbf{s}$) such that $s_n = 1$ for $n \geq N$. For $\varepsilon > 0$ and a given $\mathbf{s} \in \{-1,1\}^{\mathbb{N}}$, let $\mathbf{s}^\varepsilon \in D$ be defined by $s^\varepsilon_n = s_n$ for $n = 1, \dots, N$, and $s^\varepsilon_n = 1$ for $n > N$, where $N > -\log_2(\varepsilon)$. Clearly $d(\mathbf{s}^\varepsilon, \mathbf{s}) < \varepsilon$.

6. We know that the σ-algebra $\mathcal{S}$ is generated by all cylinders. Since cylinders are open under d (according to (3)), $\mathcal{S} \subseteq \mathcal{B}(S)$.

   On the other hand, $\mathcal{B}(S)$ is generated by open sets and each open set can be written as a union of open balls.

   Claim: Each open ball $B(\mathbf{s}, r) = B(\mathbf{s}, 2^{-n})$ is exactly the same as the open ball of the same radius, but centered on any element of D (from (5) above) which coincides with $\mathbf{s}$ on the first n coordinates.

   Proof: Follows directly from the ultra-metric property (1.3). Indeed, in an ultra-metric space, each open ball is centered around each one of its points.

   Therefore, each open set can be written as a union of open balls with centers in D and radii in $\{2^{-1}, 2^{-2}, \dots\}$. There are only countably many such open balls, so each open set can be written as a *countable* union of open balls. Thanks to the axioms of the σ-algebra, $\mathcal{B}(S) = \sigma(\text{open balls})$. We are done now, because each open ball is a cylinder, and so $\mathcal{B}(S) \subseteq \mathcal{S}$.

Problem 1.3. Let $(S, \mathcal{S})$ be a measurable space.

1. For $f, g \in \mathcal{L}^0$ show that the sets

$$\{f = g\} = \{x \in S : f(x) = g(x)\} \text{ and } \{f < g\} = \{x \in S : f(x) < g(x)\}$$

   are in $\mathcal{S}$.

2. Let $f : S \to \mathbb{R}$ be a Borel-measurable function. Show that the graph

$$G_f = \{(x, y) \in S \times \mathbb{R} : f(x) = y\},$$

   of f is a measurable subset in the product space $(S \times \mathbb{R}, \mathcal{S} \otimes \mathcal{B}(\mathbb{R}))$.

Solution:

1. It suffices to note that $\{f = g\} = F^{-1}(0)$ and $\{f < g\} = F^{-1}((-\infty, 0))$, where F is the (measurable) function given by

$$F(x) = \arctan(f(x)) - \arctan(g(x)),$$

   where $\arctan : [-\infty, \infty] \to \mathbb{R}$ extends the usual arctan from $(-\infty, \infty)$ to $\bar{\mathbb{R}}$ by continuity.

2. The coordinate maps π_x and π_y, given by $\pi_x(x, y) = x$ and $\pi_y(x, y) = y$ are measurable from $\mathcal{S} \times \mathbb{R}$ into $\mathcal{S}$ or $\mathbb{R}$. Since f is measurable, so is the composition $f \circ \pi_x : \mathcal{S} \to \mathbb{R}$. The "pair map"

$$G : \mathcal{S} \times \mathbb{R} \to \mathbb{R}^2, \; G = (f \circ \pi_x, \pi_y),$$

   is, therefore, also measurable (as a map into the product, both of whose components are measurable).

On the other hand, consider the map

$$F : \mathbb{R}^2 \to \mathbb{R},\ F(y_1, y_2) = y_2 - y_1.$$

It is continuous, and therefore, measurable. Finally, set

$$H = F \circ G,\ H : \mathcal{S} \times \mathbb{R} \to \mathbb{R}.$$

As a composition of two measurable maps, it is, itself, measurable. On the other hand

$$G_f = \{(x, y) \in \mathcal{S} \times \mathbb{R} \,:\, f(x) - y = 0\} = \{(x, y) \in \mathcal{S} \times \mathbb{R} \,:\, H(x, y) = 0\},$$

so it is measurable in $\mathcal{S} \times \mathbb{R}$.