In case the CW complex \( X \) is obtained from a subcomplex \( A \) by attaching a single cell \( e^n \), describe exactly what the extension of a homotopy \( f_t : A \to Y \) to \( X \) given by the proof of Proposition 0.16 looks like. That is, for a point \( x \in e^n \), describe the path \( f_t(x) \) for the extended \( f_t \).

**Proof.** As explained in prop 0.16 it will be enough to show an extension from a homotopy, \( F(x,t) \) on \( \partial D^n \times I \) to \( D^n \times I \). The difficulty will be in understanding what point the radial retract sends to. If we can do this we are done because the extension \( \bar{F}(x,t) = F(r(x,t)) \) where \( r(x,t) \) is the radial retract.

To start we may consider the space \( D^n \times I \) as embedded in \( \mathbb{R}^{n+1} \) and wlog having base \( D^n \) centered on the origin. Fix a point above \( D^n \times I \) a the coordinate \((0, \ldots, 0, h)\) and for any given point in \( D^n \times I \) we may uniquely create the vector \( \vec{x}_1 \) which is a vector starting at the fixed radial point and ending at the given point in \( D^n \times I \). Our goal will be to say what the homotopy extension sends the given point to. Let time \( t_1 \) be the height of the given point i.e. the last coordinate and say that in the radial retraction our original point gets sent to a point, \( x_2 \), that can similarly be described by a vector on the same line as \( \vec{x}_1 \), call it \( \vec{x}_2 \) and say the new point’s height a.k.a. last coordinate is \( t_2 \).

We will now work to describe \( \vec{x}_2 \) and \( t_2 \) as functions of \( x_1, t_1 \). To do this, first let \( \hat{x} \) be the projection of a vector \( \vec{x} \) onto it’s first \( n \) coordinates. It should be clear that \( \hat{F}(x_1, t_1) = F(x_2, t_2) \). Then using similar triangles note that \( \frac{h-t_1}{\|x_1\|} = \frac{h-t_2}{\|x_2\|} \). I’ll break this into two cases depending on whether \( x_2 \in \partial D^n \times I \) or \( D^n \times \{0\} \), that is when \( \frac{h-t_1}{\|x_1\|} \leq h \)
or $\frac{h-t_1}{\|x_1\|} > h$.

**Case:** $x_2 \in \partial D^n \times I$

In this case we know that $\|\hat{x}_2\| = 1$, so we can determine that $t_2 = \frac{\|\hat{x}_2\|}{\|\hat{x}_1\|} = \frac{h - t_1}{\|x_1\|}$. Next we can determine the point $x_2$ by also using similar triangles. I got that $x_2 = (0, \ldots, 0, h) + \vec{x}_1 \cdot \frac{\|\hat{x}_2\|}{\|\hat{x}_1\|}$.

Then homotopy $\bar{F}(\hat{x}_1, t_1) = F(\hat{x}_2, t_2)$ where $(x_2, t_2)$ are now determined.

**Case:** $x_2 \in D^n \times \{0\}$

In this case $t_2 = 0$ and therefore $\|\hat{x}_2\| = h \cdot \frac{\|\hat{x}_2\|}{h - t_1}$. Once again we can say that $x_2 = (0, \ldots, 0, h) + \vec{x}_1 \cdot \frac{\|\hat{x}_2\|}{\|\hat{x}_1\|}$. Then we can now say the homotopy $\bar{F}(x_1, t_1) = F(\hat{x}_2, 0)$.

1.1.1

Show that composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.

First I'll show too claims.

**Claim:** Given paths $a_0 \simeq a_1$ and $\beta_0 \simeq \beta_1$ and $a_i \cdot \beta_i$ i.e. the path composition makes sense, then $a_0 \cdot \beta_0 \simeq a_1 \cdot \beta_1$

This can be done by simply doing each homotopy on each part.

**Claim:** The paths $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$

The difference between the LHS and the RHS is in how much time is spent running each path, but a clear homotopy can be made for this by scaling how much time is spent running each piece of the path.

**Proof:** Given $g_0 \simeq g_1$, we can say that $\overline{g_0} \simeq \overline{g_1}$. By using the previous two claims we can concatenate paths to say $(f_0 \cdot g_0) \cdot \overline{g_1} \simeq (f_1 \cdot g_1) \cdot \overline{g_1}$ and then we get $f_0 \cdot (g_0 \cdot \overline{g_0}) \simeq f_1 \cdot (g_1 \cdot \overline{g_1})$ and this implies $f_0 \simeq f_1$.

1.1.3

For a path-connected space $X$, show that $\pi_1(X)$ is abelian $\iff$ all base point-change homomorphisms $\beta_h$ depend only on the endpoints of the path $h$.

**Proof.** $\Rightarrow$: Given two paths $f, g$ which start and end at the same points, $x_0, x_1$ respec-
tively I’ll show that $\beta_f = \beta_g$. Let $a \in \pi_1(X, x_0)$ be a loop and note the following

$$\beta_f(a) = f \bar{a} f$$

$$\simeq f(\bar{g}g)a(\bar{g}g)\bar{f}$$

$$\simeq (f\bar{g})(g\bar{g})(g\bar{f})$$

$$\simeq (g\bar{g})(f\bar{g})(g\bar{f}) \text{ by abelian assumption}$$

$$\simeq g\bar{g}$$

$$= \beta_g(a)$$

$\implies$: (Credit for this proof goes to Alisha - another student in the class whose proof I shamelessly copied). Let $f, f' \in \pi_1(X, x_0)$ be loops in the fundamental group. Now let $h$ be a path from base point $x_0$ to base point $x_1$. Using the assumption that homomorphisms are equal see that

$$hf'h = \beta_h(f') = \beta_{hf}(f') = (hf)f'(hf)$$

Therefore

$$hf'h \simeq (hf)f'(hf) \simeq hf'f'h$$

By the cancellation law of problem 1.1.1 we can say

$$f' \simeq ff'f \implies ff' \simeq ff'$$

$\square$

1.1.5

Show that for a space $X$, the following three conditions are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Proof. (a) $\implies$ (b)

Credit for this idea goes to Adrian, another student who I worked on this hmwk. with.

Let $f : S^1 \to X$ be homotopic to a constant map. Let $F_1(s)$ be the homotopy for $s \in S^1$. We will extend the map $f$ by using this homotopy. Without loss of generality say that the circle $S^1$ is a circle in $\mathbb{R}^2$ centered at the origin. Then note that every point $y$ in the $D^2$ can be described by $\frac{y}{\|y\|}(1 - t)$ for some $t \in I$ (this can be understood by thinking about the retract from $D^2$ to the origin). Then see that $\frac{y}{\|y\|} = s$ for some $s \in S^1$. Thus every point in $D^2$ can described by a tuple $(s, t) \in S^1 \times I$. Thus we can define a map $fD^2 \to X$ as $\bar{f}((s, t)) = F_1(s)$. Note that this is a cts map since $F_1(s)$ is cts over
$S^1 \times I$ and $\tilde{f}$ extends $f$ because any point in $S^1$ may be described as a tuple $(s, 0)$ so that $\tilde{f}((s, 0)) = F_0(s) = f(s)$.

(b) $\implies$ (c)
Fix an arbitrary $x_0 \in \mathbb{S}$ as a base point. Let $f \in \pi_1(X, x_0)$ be a loop. Since there is a cts map from $S^1 \to [0, 1]$, (for example by taking a trig function on $S^1$ and scaling it and adding 1) we may say $\exists f' : S^1 \to X$ s.t. $f'(s^1)$ equals the set of points in the loop of $f$. Now by part (b) extend $f'$ to a map $\tilde{f}' : D^2 \to X$ and consider $S^1$ embedded in $D^2$. Let $s_0$ be the point in $S^1$ s.t. that $f'(s_0) = x_0$ and notice that we can deformation retract $S^1$ radially through $D^2$ to the point $s_0$ since $D^2$ is a convex set. Call this deformation retract $r_t(s)$ for $s \in S^1$. Now take the composition of $\tilde{f}' \circ r_t$ and see that this is a homotopy of paths between $f$ and the trivial loop because it fixes the point $x_0$, it is continuous over $S^1 \times I$, and at time 0 the map is $f(S^1)$ while at time 1 it is simply the point $f(s_0)$ which is the trivial loop. Thus $[f] = 0 \in \pi_1(X, x_0)$.

(c) $\implies$ (a)
This implication is clear. Let $f : S^1 \to X$ be any cts map and fix a point $s_0 \in S^1$. Then let $x_0 = f(s_0)$. Since there is a map from $[0, 1]$ to $S^1$ we may regard $f$ as a loop starting and ending at $x_0$. By the assumption of (c), $\pi_1(X, x_0) = 0$ so there exists a homotopy between $f$ and the trivial map, call it $F_t$. Then by considering the point $x_0$ as the image of a constant map on $S^1$ sending everything to $x_0$ we see that $F_t$ is a homotopy between cts maps $f$ and a constant map.

Deduce that a space $X$ is simply-connected i all maps $S^1 \to X$ are homotopic. [In this problem, homotopic means homotopic without regard to basepoints.]

**Proof.** This follows from the above equivalent statements. $X$ is simply connected $\iff$ $\pi_1(X, x_0) = 0$ for any $x_0$ $\iff$ Every cts map from $S^1 \to X$ is homotopic to a constant map $\implies$ all maps from $S^1 \to X$ are homotopic. Actually this needs some kind of condition on the maps from $S^1$. They need to have the same base point, and then they will be homotopic.

1.1.6

We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that $\phi$ is onto if $X$ is path-connected, and that $\phi([f]) = \phi([\bar{g}])$ if $[f]$ and $[\bar{g}]$ are conjugate in $\pi_1(X, x_0)$. Hence $\phi$ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when $X$s is path-connected.

Claim: $\phi$ is onto if $X$ is path-connected.
Proof. Let \( f \in [S^1, X] \) be a cts map. Since there is a cts map from \([0, 1] \) to \( S^1 \) we may compose this map \( f \) and consider \( f \) as some path in \( X \). Using the path connected property of \( X \) we may say there exists a path \( h \) from \( s_0 \) to \( x_0 \). Then we may define a new map \( h_t(s) : [0, 1] \) s.t. \( h_t = (st) \) i.e. \( h_t \) only travels part of the path of \( h \). We may now define a map \( \psi_t = h_t \circ f \circ h_t \), by traveling each piece for \( 1/3 \) of the time. This \( \psi_t \) is cts over \([0, 1]\) (which we may regard as \( S^1 \)) and \( \psi_0 = f \) while \( \psi_1 \) is a path from \( x_0 \) returning to \( x_0 \) i.e. it is an element in \( \pi_1(X, x_0) \). This shows that the map \( \phi \) is surjective. \( \qed \)

Claim: \( \phi([f]) = \phi([g]) \iff [f] \text{ and } [g] \text{ are conjugate in } \pi_1(X, x_0) \).

Proof. \( \implies \): I got the idea for this from a posting on Math Stack Exchange. We apply Lemma 1.1.19. Assuming that \( f, g \in \pi_1(X, x_0) \) and that \( \phi([f]) = \phi([g]) \) then there exists a homotopy \( \psi_t : S^1 \to X \). By looking at \( \psi_t(s_0) \) we get a path, I’ll call \( h \). In fact, since \( f(s_0) \) and \( g(s_0) \) are assumed to be \( x_0, h \) is a loop in \( \pi_1(X, x_0) \). Now we may apply Lemma 1.1.19 to say that \( \psi_{0*} = \beta_h \psi_{1*} \) which shows that \( [g] = [h] \cdot [f] \cdot [h] \). This proves that \([f]\) and \([g]\) are conjugates in \( \pi_1(X, x_0) \).

\( \iff \): Assume that the maps \( f, g \in \pi_1(X, x_0) \) are conjugate to each other by a loop \( h \in \pi_1(X, x_0) \) so that \( [hf\bar{h}] = [g] \). Then under the map \( \phi \) we get that \([hf\bar{h}] = [g]\) as equivalence classes in \([S^1, X]\). All that remains is to show that \([hf\bar{h}] = [f]\) as elements in \([S^1, X]\), but this is true by using a similar homotopy that was described in the proof that \( \phi \) is surjective for a path connected space. Therefore by transitivity we get that \([f] = [g]\) as equivalence classes in \([S^1, X]\) and therefore \( \phi([f]) = \phi([g]) \). \( \qed \)

The conclusion of the one-to-one correspondence clearly follows from the above statement.

1.1.8

Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map \( f : S^1 \times S^1 \to \mathbb{R}^2 \) must there exist \((x, y) \in S^1 \times S^1 \) such that \( f(x, y) = f(x, y)\)?

Proof. I got a hint for this from a solutions manual from Ohio University which was that this statement is not true. **Note to self: The idea here was to center the torus at the origin in \( \mathbb{R}^3 \) and think about where the antipodal points were.**

Consider the torus in \( \mathbb{R}^3 \) centered at the origin. Then consider the map that is the projection onto the \( XY \) plane. The projection is always a continuous map. Additionally for any point \( p \) on the torus the map \( \pi(-p) = -\pi(p) \). Therefore this is a counterexample to the Borsuk-Ulam theorem for the torus. \( \qed \)
1.1.12

Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism $\phi$ of a map $\phi: S^1 \to S^1$.

Proof. The fundamental group of $S^1 \cong \mathbb{Z}$ so any homomorphism $\phi: \mathbb{Z} \to \mathbb{Z}$ is determined by where $\phi$ sends 1. Given a homomorphism $\phi \ast$ s.t. $\phi \ast(1) = n$, I will show that there exists a map which sends one loop around $S^1$ i.e. the elt. 1, to the elt. of $n$ loops around $S^1$. Without loss of generality, say the circle $S^1$ is embedded in the complex plane and the base point corresponds. Then define a map $\phi(e^{i2\pi\theta}) = e^{i2n\pi\theta}$. This map will clearly send one loop to $n$ loops since as $\theta$ ranges from 0 to 1, $e^{i2n\pi\theta}$ will loop around $S^1$ once but the function $e^{i2n\pi\theta}$ will loop around $S^1$ $n$ times. By construction this map $\phi$ will clearly induce the desired homomorphism.

1.1.13

Given a space $X$ and a path-connected subspace $A$ containing the basepoint $x_0$, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \to X$ is surjective if every path in $X$ with endpoints in $A$ is homotopic to a path in $A$.

Proof. $\implies$: The idea for this comes from a solutions manual from bard college.

**Note to self : I could easily come up with a map homotopy, simply connect $x_0$ to the endpoints by a path, but I wasn’t sure how to make this a path homotopy. I think the idea for this one was to make chains of path homotopy “equalities” **

Let $f$ be a path in $X$ with end points in $A$. Using the path connectedness of $A$ we may say there exists paths $a, b$ from a respective endpoint of $f$ to the point $x_0$. Then note that the path $\bar{a}fb$ is a loop in $\pi_1(A, x_0)$ and by assumed surjectivity of $i_*$, there exists a loop $g$ which is homotopic to the loop $\bar{a}fb$. Now we may have a path homotopy from $a(\bar{a}fb)b$ to $agb$ by applying the map homotopy only to center piece of this path. By leaving the outer paths fixed this will be a path homotopy. Now we can make the following chain of path homotopies.

$\quad f \simeq a(\bar{a}fb)b \simeq agb$

This last path $agb$ is a path contained in $A$. This completes the forward direction.

$\iff$: This direction is pretty trivial. Given that a loop $f \in \pi_1(X, x_0)$ this is a path with endpoints in $A$ therefore there is a path $f' \in A$ which is path homotopic to $f$. By this path homotopy the class of loops $[f] = [f']$ in $\pi_1(X, x_0)$ so that $i_*([f]) = [f]$ and we have shown surjectivity.

□